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A parametric solution for simple stress–strength model of failure with an application

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Abstract: Estimation of $P(R \leqslant S)$ is considered for the simple stress–strength model of failure. Using the Pareto and Power distributions together with their combined form a useful parametric solution is obtained and is illustrated numerically. It is shown that these models are also applicable when only the tails of distributions for R and S are considered. An application to the failure study concerning the fractures is also included

Keywords: Stress-strength, failure, parametric, Power, Pareto, tail, reliability, fracture.

1. Introduction

The problem of estimating the reliability for the so-called stress–strength model of failure has been considered by a relatively large number of investigatores (see e.g. [17,31] and reference therein). Let R and S be two random variables with respective distribution functions $F_R(\cdot)$ and $F_S(\cdot)$. Suppose R is the strength (resistance) of a component (or a structural element) subject to a stress S. Then the structural element (component) fails if at any moment the applied stress (or load) exceeds the strength. The stress is a function of the environment to which the structure is subjected. Strength depends on material properties as the main factor and also manufacturing procedures and so on. The reliability of a structural element (or component) is therefore

$$P(R > S) = 1 - P(R \leqslant S) = 1 - P_f = \int_0^\infty F_S(x)\,\mathrm{d}F_R(x)$$

where P_f is the probability of failure due to a single application of the load (stress).

The stress-strength model introduced above has been considered by Birnbaum [4] for the first time and later found an increasing number of applications in many different areas especially in the structural engineering. For a bibliography of available results see [1,2,17,23,31,33].

Now, when applying the above model one is frequently interested in the reliability of the structural element (or component) for a specified interval, say $(0,t]$. If the life of the structure is measured in time T, the probability of failure, denoted by $F_T(t)$ in time interval $(o,t]$ is measured by

$$F_T(t) = P(T \leqslant t) = 1 - P(T > t) = 1 - L_T(t)$$

where $L_T(t)$ is survival (or reliability) function defined as $L_T(t) = P(T > t)$, $L_T(0) = 1$. If R is the strength of a structural element subjected to a sequence of stresses $S_1, S_2, \ldots$, then $L_T(t)$ is given by

$$L_T(t) = \sum_{r=0}^{\infty} P(N(t) = r)\bar{P}(r) \tag{1}$$

where $\{N(t),\ t \geqslant 0\}$ is a general counting process of stresses occuring randomly in time and

$$\bar{P}(r) = P[\max(S_1, S_2 \ldots S_r) < R] \quad (\bar{P}(0) = 1), \quad r = 1, 2, \ldots.$$

An example of interest is the case where a structure such as a nuclear power plant is subjected to a sequence of stresses $S_1, S_2 \ldots$ caused by natural events such as earthquakes or storms. The model (1) which has recently been considered by Ebrahimi [13] is a special case of the model studied by Esary et al. [14]. If as in the latter reference we assume that the occurrence of the stresses is governed by a Poisson process (this is a usual assumption in probabilistic design) then we have

$$L_T(t) = \sum_{r=0}^{\infty} \left[e^{-\lambda t}(\lambda t)^r / r!\right] \bar{P}(r). \tag{2}$$

Note that here $\bar{P}(r)$ can be considered as the probability of surviving the first r shocks. If we further assume that the stresses are independent and identically distributed random variables, from (2) we get

$$L_T(t) = \sum_{r=0}^{\infty} \left[e^{\lambda t}(\lambda t) r / r!\right](1 - P_f)^r = \exp(-\lambda t P_f). \tag{3}$$

This is the function which has often been used in the engineering context (see e.g. [32]).

Now as can be seen from (3), if the mean rate of stress occurrence and the life time of the structure are given then $L_T(t)$ can be calculated for any P_f. Thus the main problem for the situation described above is that of estimating the P_f, that is the probability of failure due to single application of the load (stress). Considering this the main object of this article is to present a simple method for estimation of $P_f = P(R \leqslant S)$ and provide justification for that. Here, consideration will be given to the tails and extremes which are important factors for the design of structures. Some practical problem of engineering will also be included for demonstration.

2. On estimation of $P_f = P(R \leqslant S)$

In most of the studies concerning the stress–strength model for failure it is assumed that the distribution of S (or of both S and R) will be completely known except possibly for a few unknown parameters and it is desired to obtain parametric solutions. Reiser and Guttman [31] Church and Harris [6], Owen et al. [28] and Govindarajulu [18] have considered the above problem under the assumption that S and R have normal distribution (see also [24]). Basu [2] has used distributions such as exponential and gamma which are useful in life testing. Basu and Ebrahimi [3] have extended some of the existing results to the case where stress and strength form a certain kind of stochastic processes. Some nonparametric results are also available for confidence bounds and especially for the upper bound (see e.g. [18]) based on statistics U defined to be the number of pairs $(R_{(i)}, S_{(j)})$ such that $R_{(i)} < S_{(j)}$. Here $R_{(1)} \geqslant R_{(2)} \geqslant \ldots \geqslant R_{(n)}$ and $S_{(1)} \geqslant S_{(2)} \geqslant \cdots \geqslant S_{(m)}$ are order statistics corresponding to samples of independent observations of sizes n and m respectively. In fact using the above statistics and unbiased estimator of P_f is U/mn. It is worth mentioning that a similar statistics which was proposed by Prochan and Sullo [30] is used in [2] for estimation of the reliability under the assumption that R and S have a bivariate exponential distribution.

Now in the most of the references mentioned above only the theoretical aspects of the problem were of interest. It is clear that except for very special cases calculation of P_f would need numerical integration even if the distributions are known. On the other hand assumptions such as normal or exponential distributions for both stress and strength which are easy to handle may have a limited value in acual practice.

Indeed when considering the random factors for a design problem, loads with high intensity (e.g large earthquakes or high-speed winds) and components (materials) with low resistance are of concern. In other words, whatever the forms of distributions for R and S we are mainly interested in their tail behaviour, that is, the range of low values for R and high values for the S and of course their interaction. Since, in general, $F_R(\cdot)$ and $F_S(\cdot)$ are unknown and could take a large number of different forms one can following the above approach (tail consideration) reduce the problem to a set containing a small number of parametric families. This is because, it is shown that some large classes of arbitrary distributions have a similar tail behaviours. Thus following this approach we may calculate P_f for the possible acceptable combination.

Finally since there might be cases where P_f has to be calculated (estimated) based on complete distribution, we will first present an approximation for unimodal distributions which will enable us to find P_f explicitly.

3. A practical method for estimation of P_f

Since in practice the distribution of R and S are not completely known, engineers had often used the upper bound of P_f. The usual upper bound considered was the one obtain from the so-called Camp–Meidell inequality. This of course, gives a conservative solution. However in recent years due to the high rise in cost of the structures some investigatores have looked for a less conservative solution for the problems of this kind. See [10] and references there in. An important attempt in this direction was made by Wirsching [36] who studied the behavior of the statistical models which are usually used for design. He had considered eight two-parameter statistical models which are often used in design and compaired their tail probabilities with each other and also with the result obtained from Camp–Meidell inequality assuming that the first two moments are known. He then selected the exponential and power models for quasi-upper bounds of right and left tails respectively. (Note that this is different from the approach involving the tails refered to in Section 2). The P_f corresponding to the above choice is

$$P_f = P(R \leqslant S) = \exp(-\lambda(b-a)) + \frac{\lambda}{b^\alpha}\int_a^b x^\alpha \exp(-\lambda(x-a))\,\mathrm{d}x \tag{4}$$

Table 1

R	0.0352	0.0397	0.0677	0.0233	0.0873
S	1.77	0.9457	1.8985	2.6121	1.0929
R	0.1156	0.0286	0.0200	0.0793	0.0072
S	0.0362	1.0615	2.3895	0.0982	0.7971
R	0.0245	0.0251	0.0469	0.0838	0.0796
S	0.8316	3.2304	0.4373	2.5648	0.6377

where

$$F_R(x) = (x/b)^\alpha, \quad 0 \leqslant x \leqslant b, \quad \text{and} \quad F_S(x) = 1 - \exp(-\lambda(x-a)), \quad x > a.$$

In an later attempt Dargahi-Noubary [7] following the same lines and using the fact that results concerning the upper tail probabilities of a distribution function for X can be deduced from those for lower tail of distribution function for $y = 1/x$, has found that the pairs Power and Pareto is a 'better' choice. For this selection the distribution functions and the corresponding P_f are respectively (all parameters are positive)

$$F_R(x) = (x/b)^\alpha, \quad 0 \leqslant x \leqslant b, \qquad F_S(x) = 1 - (a/x)^\beta, \quad x \geqslant a,$$

and

$$P_f = \frac{1}{\alpha - \beta}\left[\alpha\left(\frac{a}{b}\right)^\beta - \beta\left(\frac{a}{b}\right)^\alpha\right]. \tag{5}$$

Note that here (unlike (4)) no numerical integration is required. It is worth mentioning that (5) can also be written in terms of central safety factor and coeffficient of variation (signal-to-noise ratio) which are design guides familiar to the engineers (see [7]). Also in the above study the second 'best' pairs is found to be the Weibull–Frechet which had been introduced by Freudenthal [15] and used by several investigators.

Before proceeding further let us consider the following example studied in [2] and [12]. In this example a random samples of fifteen pairs of (R, S) values are drawn (see Table 1). Estimates for P_f under the certain assumptions concerning the distribution R and S are given in [2]. Assuming exponential for both strength and stress P_f is found to be 0.0361. Assuming gamma for R and S the following estimates are reported for three different sets of values for the parameters; $\hat{P}_f = 0.0361$, 0.0048 and 0.0038. Assuming normal distribution for S and IFA and IFRA for R the values $\hat{P}_f = 0.067$ and 0.097 are found in [12]. For this examples the estimate of P_f using the statistics U introduced in Section 2 is 0.04. Also P_f corresponding to the Pareto–Power choice is 0.0829.

Now, it is mentioned in [2] that the P_f value known from the past records in 0.05. If we assume that the true value is around 0.05 then (note that, in this example we have one case of $R < S$, so that $\hat{P}_f \approx 0.067$) it is clear that the results based on gamma and exponential are all underestimation. This, as was mentioned before, would not appreciated by engineers. The results obtained from Power–Pareto is, of course, conservative as expected.

Turning to the problems in acutal practice we note that for assessment of reliability using parametric approach it often difficult to distinguish which of several competing models provides the best description. In fact, as is also demonstrated above, the choice of model has, in most

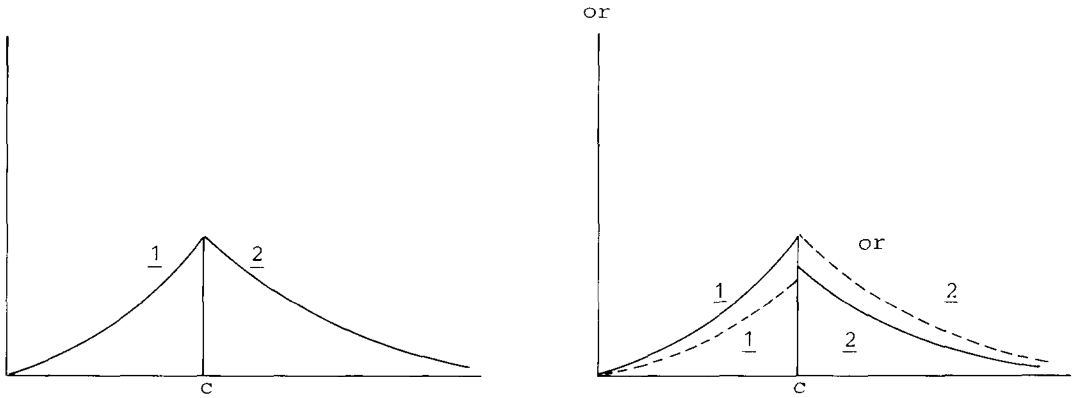

Fig. 1. Illustration of Power and Pareto distributions

cases, a profund effect on probability estimates particularly in the tails of distribution see e.g. [27] for further demonstration.

Now considering these difficulties one simple solution would be to consider an approximation which replaces the distribution involved by the one of the form shown in the Fig. 1. (Note that the random variables representing R and S are assumed to be non-negative). Here (1) and (2) are taken to be the Power and Pareto distributions respectively. If the original distribution is unimodal, then for a wide class of distribution this would give a good approximation especially in the tails. One advantage of such approximation is that no change in small values of, for example, stress (e.g. values to the left of median or mode) will have any effect on the right tail part which is the region of interest and importance. In fact, when fitting an ordinary unimodal distribution (e.g. exponential) to the stress data a small adjustment for values close to the lower bound may have a significant effect on prediction of the probabilities of the future large values and hence P_f. This point will also serve as a reason to consider the tails rather than the whole distribution. Other advantages of such approximation are the following:

(a) Given C (see Fig. 1), one only requires to estimate α and β. For example c may be taken to be the mean (median) or the mode of the all available data.
(b) Likelihood estimates of α and β are simple to obtain and no numerical calculation is involved. Also distribution of these estimates are known and can be used for inference.
(c) It includes the case where only one of the two distributions may be fitted to the whole data to provide a conservative answer as discussed before.
(d) Interaction of two such distributions (approximations) which is required for estimation of P_f, does not involve any numerical integration.
(e) Accepting the usual assumption of Weibull distribution for the strength, the condition

$$\lim_{t \to 0} \frac{F(\mu + tx)}{F(\mu + t)} = x^{\alpha} \quad \text{for all} \quad x > 0$$

which is necessary and sufficient for the Weibull limiting distribution of sample minima, supports selection of the power distribution for the lower tail.

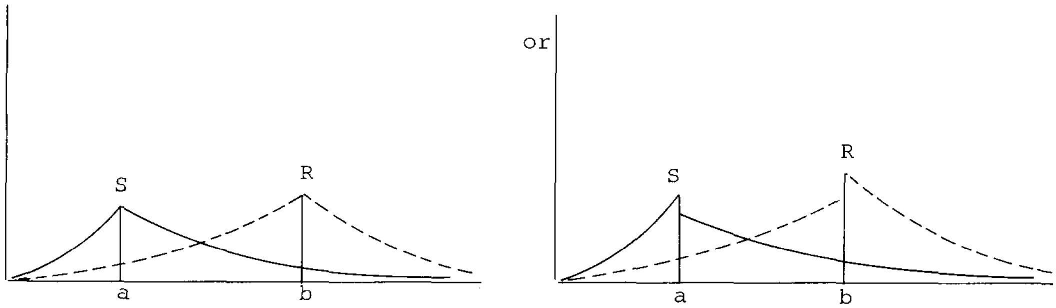

Fig. 2. Combined Power–Pareto distributions for R and S

Considering (d) suppose that we have

$$F_R(x)=\begin{cases}F_R^{(\mathrm{L})}(x), & 0\leqslant x\leqslant b,\\ F_R^{(\mathrm{U})}(x), & x>b,\end{cases}\qquad F_S(x)=\begin{cases}F_S^{(\mathrm{L})}(x), & 0\leqslant x\leqslant a,\\ F_S^{(\mathrm{U})}(x), & x>a,\end{cases}$$

where

$$F_R^{(\mathrm{L})}(x)=c_1x^{\alpha_1},\qquad F_R^{(\mathrm{U})}(x)=1-d_1/x^{\beta_1},$$
$$F_S^{(\mathrm{L})}(x)=c_2x^{\alpha_2}\quad\text{and}\quad F_S^{(\mathrm{U})}(x)=1-d_2/x^{\beta 2}.$$

Then the required probability is

$$\begin{aligned}P_f&=P(R\leqslant S)\\&=\frac{\alpha_2}{\alpha_1+\alpha_2}F_R^{(\mathrm{L})}(a)F_S^{(\mathrm{L})}(a)+\frac{\alpha_1}{\alpha_1-\beta_2}\big(1-F_S^{(\mathrm{U})}(b)\big)F_R^{(\mathrm{L})}(b)\\&\quad-\frac{\beta_2}{\alpha_1-\beta_2}\big(1-F_S^{(\mathrm{U})}(a)\big)F_R^{(\mathrm{L})}(a)-\frac{\beta_1}{\beta_1+\beta_2}\big(1-F_S^{(\mathrm{U})}(b)\big)\big(1-F_R^{(\mathrm{U})}(b)\big).\end{aligned}$$

If for example, (a) and (b) are taken to be the medians of S and R respectively, then

$$P_{\mathrm{f}}=\frac{1}{2}\cdot\left(\frac{\alpha_2}{\alpha_1+\alpha_2}-\frac{\beta_2}{\alpha_1-\beta_2}\right)\cdot F^{(L)}\mathrm{L}_R(a)-\frac{1}{2}\cdot\left(\frac{\beta_1}{\beta_1+\beta_2}-\frac{\alpha_1}{\alpha_2-\beta_2}\right)\cdot\big(1-F_S^{(\mathrm{U})}(b)\big)$$

Let us now consider the case where only the tails of distributions for R and S will be involved in failure estimation. As mentioned before for a reliable structure or component the probability of failure P_{f} is small and is determined by the tails of the distributions involved. Given a set of data one may not be interested in entire statistical population, so that no assumption is made about the center of the distribution. For example in structural engineering this point is of great importance since often extrapolations have to be made for beyond the range over which data (observations) are available. Thus the predication, particularly in the range of interest will be much more reliable based on models for tails then those based on statistical fit procedures of the initial population. However although important to the best of our knowledge no attempt is made to model the tails and to use them for estimation of the P_{f}. Chen and Lind [5] have considered a three parameter normal tail approximation to a nonnormal distribution. This, as we shall shortly discuss is not a natural approximation.

Now having accepted such necessity we should then find (using some ceriterian) suitable models for the lower tail of R and also the upper tail of S. For this a satisfactory solution could be based on the largest (or smallest) K order statistics following the theory introduced in [29]. Briefly, based on this theory, for large samples the tails of all common distributions can only take one of the three possible forms (known as generalized Pareto distribution). The random variable X is said to have the generalized Pareto distribution with parameters γ and σ if $P(X > x) = (1 + \gamma X/\sigma)^{-1/\gamma}$, $-\infty < \gamma < \infty$, $\sigma > 0$, $x > 0$, $\gamma x > -\sigma$ (The distributions with $\gamma = 0$ are defined to be the exponential distributions with mean σ). This family of distributions has several nice properties. All manner of tail behaviors are presented. Pickands [29] has introduced a method for estimation of the unknown parameters, which also leads to selection of the statistically appropriate model. Other estimating methods including the maximum likelihood are described in [9,11,35].

Now, although the above model is very powerful the 'best' model may turn out to be different for two sets of data from the same population. Also for some possible pairs, calculation of P_f will involve numerical integration. Moreover when estimating unknown parameters the maximum likelihood method does not work for $\gamma < \frac{1}{2}$.

Noting that in practice one is interested in simplicity as well as the accuracy and this may not be achieved without a compromise we suggest application of Pareto and its 'inverse' power for the upper tail and lower tail of S and R respectively. There are also some justifications for these approximations. First we refer to the generalized Pareto distributions and note that when $\gamma > 0$ we have

$$P(X > x) \approx \mathrm{d}x^{-1/\gamma}$$

Second we recall the comparison made in [36] and followed in [7] which led to selection of Pareto–Power as the 'best' pairs for S and R. Also as mentioned before the result of above mentioned study showed that the second 'best' pairs to consider is Frechet–Weibull which had been introduced by Freudenthal [15] who developed what is now called a classical structural reliability theory (see also [34]). This shows that the Pareto–Power is also a good approximation for this choice. Third as is pointed out in [17] when estimating the reliability, traditionally S and R are assumed to be independent lognormal random variables. To clearify these we refer to the fact that the distributions mentioned for S are all heavytailed distributions for which

$$P(X > x) \approx \mathrm{d}x^{-\beta} \quad \text{as } x \to \infty \tag{6}$$

is a natural assumption or approximation. For example it is shown in [11] that the tail of a log-normal distribution is Pareto with $\gamma = 0.259$. Other references which have considered (6) are [19,20,21,22]. For strength the Weibull distribution is generally accepted. It is fairly evident that if X has a Frechet distribution, then X^{-1} has a Weibull distribution. Since the same relation holds between Pareto and Power distributions it is natural to model the lower tail of the R by the latter distribution.

Turning to the advantages from inference point of view we note that Hill [22] has proposed a simple general approach to make inference about the tail behavior of a distribution. This is an approach which does not require any assumption concerning the global form for the distribution function, but merely the form of behavior in the tail where it is desired to draw inference. His study has shown that the inference for the upper tail are particularly simple if the model (6) is assumed (see below).

Let $X_1, \ldots, X_n$ be a sample of size n on a positive random variable with distribution function $F(x) = 1 - d_1 x^{-\beta}$ for $x \geqslant a$ where a is known. Then conditioning upon the values $x_{(i)}$, $i = 1$,

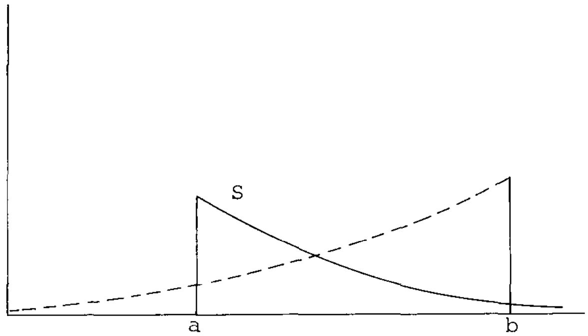

Fig. 3. Power/Pareto tail models for R and S

2,..., $(r+1)$ of the $r+1$ largest observations where $x_{(r+1)} \geqslant D$, Hill has found the following maximum likelihood estimates for the β and d,

$$\hat{\beta} = (r+1) \Big/ \left[\sum_{i=1}^{r} \log(X_{(i)}) - r \log X_{(r+1)} \right],$$

$$\hat{d} = \left[X_{(r+1)} \right]^{\hat{\beta}} (r+1)/n.$$

Also, for Power distribution namely $F(x) = cx^{\alpha}$ for $x \leqslant b$ where b is known he has obtained (conditioning on $X_{(n-r)} \leqslant d$)

$$\hat{\alpha} = [(r+1)/r] \left[\log X_{(n-r)} - r^{-1} \sum_{i=0}^{r-1} \log X_{(n-i)} \right]^{-1},$$

$$\hat{c} = [(r+1/n] \left[x_{(n-r)} \right]^{-\hat{\alpha}}.$$

The asymptotic normality of these estimates is investigated in Haeusler and Teugels [21] which if required can be used to constract confidence intervals.

Finally since the method depend upon a subjective choice of the threshold Hill has described some data-analytic techniques which can be useful in the choice of r upon which to base the analysis.

Turing to calculation of failure probability (P_{f}) using the proposed tail models, let

$$F_R(x) = cx^d, \quad 0 \leqslant x \leqslant b, \qquad F_S(x) = 1 - d/x^{\beta}, \quad x \geqslant a,$$

then it can be shown that

$$P_f = P(R \leqslant S) = \frac{cd}{\alpha - \beta}(\alpha b^{\alpha-\beta} - \beta a^{\alpha-\beta})$$

$$= \frac{\alpha}{\alpha - \beta}(1 - F_S(b))F_R(b) - \frac{\beta}{\alpha - \beta}(1 - F_S(a))F_R(a). \tag{7}$$

If rather than tails we consider whole the data and fit Power and Pareto distribution to them, then since for this case we have $F_S(a) = 0$ and $F_R(b) = 1$, (7) will reduce to

$$P_f = \frac{\alpha}{\alpha - \beta}(1 - F_S(b)) - \frac{\beta}{\alpha - \beta} F_R(a)$$

which is same as (5).

Finally, to use the above models to estimate the tail behaviour from sample observations on a variable X it would be necessary to choose a value X_0, say, beyond which data can be used for estimation of the parameters. If this is not known then we may following DuMouchel [11] take x_0 to be the 90th (or 10th) percentile of the sample provided that the size of sample is not small. An alternative is to use one of the data-analytic techniques described in [22] and increase r step by step. Note that, in general, physical considerations specific to the subject matter are pertinent and so it is impossible to give more than rather general guidelines. Here a threshold level with a direct physical interpretation should be chosen where possible, provided of course that a useful model results.

Example 1. Consider the data presented in Table 1 and the order statistics corresponding to stress S and strength R. Clearly with such small sample sizes we can not talk about the tails. However to demonstrate the effect that centeral values could have on tails and consequently on P_f we have ruled out the smallest sample value 0.0072 corresponding to S and the largest sample value 3.2304 corresponding to R. The resulting P_f, after fitting the tail models and using (7) is 0.043 which is close to 0.05 and has a significant differences with 0.083 based on complete samples.

Example 2. This example considers the sample on S to be the observed yearly maximum of the one-hour mean in wind in London, Ontario for the year 1939–1961 (unit of speed m/s). The order statistics corresponding to this data (compiled from [8]) are:

14.3	14.8	15.2	15.6	16.1	16.1	16.5	16.5
17.4	17.4	17.4	17.4	18.3	18.3	18.8	19.2
20.1	20.1	20.6	22.4	23.2	24.6	25.9	

Davenport [8] found the Type I extreme value distribution namely $F(x) = \exp(-\exp-(x-17)/13)$ to give a reasonable fit to the data. Note that since the Rayleigh distribution has been generally used as a parent distribution for wind speed and this distribution belongs to the domain of attraction for the Type I extreme value distribution one would expect the maximum speed to follow the latter distribution. Now to fit a Pareto tail we should first choose a cutoff point. Some possible values are 18.3 which is next to the mode of sample (the mode using the above distribution is 17) or 20.3 that is the point of inflection of the sample histogram (this may also be obtained from above distribution). The distribution functions corresponding to the above values and also the one corresponding to whole data are.

$$\begin{aligned}
&(1)\quad F_1(x) = 1 - 1472166313\,x^{-7.51577}, && x \geqslant 18.3,\\
&(2)\quad F_2(x) = 1 - (1.3295292)^{10}\,x^{-8.2097}, && x \geqslant 20.6,\\
&(3)\quad F_3(x) = 1 - 48851\,x^{-4.05845}, && x \geqslant 14.3.
\end{aligned}$$

To make comparison let us, for example, consider the probability of $X > 25$. Since we have only one observation greater than 25 a direct estimate for this is $1/23 = 0.0435$. The values corresponding to (1), (2) and (3) are respectively 0.0459, 0.0444 and 0.1036. Also for $F(x)$ obtained by Davenport [8] we have 0.0671. Note once again the advantage of considering the tails rather than the whole data.

Now turning to calculation of failure probability let us (due to lack of data) make the usual assumption of constant strength for the structure to be designed, that is let $R = R_0$. R_0 is usually refered to as characteristic value in the engineering context. Then we may use the formula

$$P_f = P(S > R_0) = 1 - F_S(R_0)$$

and obtain P_f for different design values. As an example if a structure is designed to resist wind speed of 25 mls then (using model (2)) $P_f = 0.0444$ and therefore for a lifetime of 20 year we have

$$L_T(20) = \exp(-1/23 \times 20 \times 0.0444) = 0.962.$$

4. An application to fracture problems

In the failure study concerning the fracture due to crack growth experts often consider the results of the linear elastic fracture mechanics see e.g. [16]. In berif, this theory suggests that gross failure occurs when the stress intensity factor (k_1) around a crack with a depth A in a nominal stress field S exceeds the plain strain fracture toughness K_{1c} i.e. when

$$K_1 = S(M \cdot A)^{1/2} \geqslant S_c(M \cdot A_c)^{1/2} = K_{1c}$$

where M is a constant which depends upon the type of load and the geometry of the crack (in fact it is a correction for idealization geometry) and S_c and A_c are critical stress and critical defect sizes respectively. Note that rather than using the above inequality the usual approach has been to consider the equivalent formulea

$$A = \frac{1}{M}\left(\frac{K_1}{S}\right)^2, \qquad A_c = \frac{1}{M}\left(\frac{K_{1c}}{S_c}\right)^2,$$

and define P_f as

$$P_f = P(A_c \leqslant A)$$

where A und A_c are as mentioned, the actual and critical defect sizes respectively.

Before proceeding further it should be mentioned that for this problem on line of research has been to consider a crack growth model for A (see e.g. [17,25]) and use the result together with a non-random A_c to obtain an estimate for P_f. Note that this latter assumption is, in fact, hard to justify since, as is well-known, it has been found by experience that the K_{1c} of material manufactured under similar conditions show a significant stochastic variation from piece to piece. (For a statistical aspect to strength of materials see e.g. [26, ch. 14].

Turning to a practical aspect of the problem we note the critical defect size depends on fracture toughness K_{1c} and also critical stress S_c. Given data on K_{1c} and S_c one may need to use them to obtain an approximate distribution for A_c. If S_c could taken to be a non-random variable, then distribution of A_c is easy to determine from that of K_{1c}. If this is not the case derivation may pose difficulties even if the distributions of K_{1c} and S_c are known. Considering this and the fact that these distributions, in general, are not known, we may consider once more distributions of Pareto and Power type for the random variables involved. In fact since Weibull is the distribution often applied for strength of material and Frechet is the one which has been used frequently both for depth and length of defects, shows that such approximations are reasonable. It is worth mentioning that if we assume Frechet and Poisson distributions as the model for the largest defect size and number of defects respectively then it may be concluded that the initial defects size follow a Pareto distribution. Since (taking logarithms) we have

$$\log A_c = 2 \log K_{1c} - 2 \log S_c - \log M$$

some investigators using the result that any linear combination of normal random variables is also a normal variable have assumed log-normal distribution for K_{1c} and S_c resulting in log-normal distribution for A_c. As discussed before for this distribution, as well the Pareto tail is a good approximation.

Turning to derivation of A_c's distribution suppose K_{1c} and S_c are independent and have distributions of the Power–Pareto type introduced in Section 3. We consider this general form since other cases mentioned before, are its special cases. Let

$$f_{K_{1c}}(x) = \begin{cases} f_{K_{1c}}^{(L)}(x) = \alpha_1 c_1 x^{\alpha_1 - 1}, & 0 \leqslant x \leqslant b, \\ f_{K_{1c}}^{(U)}(x) = \beta_1 d_1 / x^{\beta_1 + 1}, & b < x < \infty, \end{cases}$$

and

$$f_{S_c}(x) = \begin{cases} f_{S_c}^{(L)}(x) = \alpha_2 c_2 x^{\alpha_2 - 1}, & 0 \leqslant x \leqslant a, \\ f_{S_c}^{(U)}(x) = \beta_2 d_2 / x^{\beta_2 + 1}, & a < x < \infty, \end{cases}$$

represent the density functions of K_{1c} and S_c respectively, then it is not difficult to show that

$$f_{A_c}(x) = \begin{cases} f_{A_c}^{(L)}(x) = \frac{1}{2}\alpha_1 w_1 x^{\alpha_1/2 - 1} + \frac{1}{2}\beta_2 w_2 x^{\beta_2/2 - 1}, & 0 \leqslant x \leqslant b^2/Ma^2, \\ f_{A_c}^{(U)}(x) = \frac{1}{2}\alpha_2 w_3 / x^{\alpha_2/2 + 1} + \frac{1}{2}\beta_1 w_4 / x^{\beta_1/2 + 1}, & b^2/Ma^2 < x < \infty, \end{cases}$$

where

$$w_1 = c_1(Ma^2)\alpha_1/2\left[c_2\alpha_2 a^{\alpha_2}/(\alpha_1 + \alpha_2) - d_2\beta_2 a^{-\beta_2}/(\alpha_1 - \beta_2)\right],$$
$$w_2 = d_2(Mb^{-2})\beta_2/2\left[d_1\beta_1 b^{-\beta_1}/(\beta_1 - \beta_2) + c_1\alpha_1 b^{\alpha_1}/(\alpha_1 - \beta_2)\right],$$
$$w_3 = c_2(Mb^{-2}) - \alpha_2/2\left[c_1\alpha_1 b^{\alpha_1}/(\alpha_1 + \alpha_2) - d_1\beta_1 b^{-\beta_1}/(\alpha_2 - \beta_1)\right],$$
$$w_4 = d_1(Ma^2) - \beta_1/1\left[d_2\beta_2 a^{-\beta_2}/(\beta_1 + \beta_2) + c_2\alpha_2 a^{\alpha_2}/(\alpha_2 - \beta_1)\right],$$

Note that while $f_{A_c}^{(L)}(x)$ is a mixture of Power distributions the $f_{A_c}^{(U)}(x)$ is a mixture of Pareto distributions.

Next having determined the distribution of A_c the failure probability may then be calculated as follows. Let

$$f_A(x) = \begin{cases} f_A^{(L)}(x) = \alpha c x^{\alpha - 1} & 0 \leqslant x \leqslant h, \\ f_A^{(U)}(x) = \beta d / x^{\beta + 1} & h < x < \infty, \end{cases} \qquad F_A(x) = \begin{cases} F_A^{(L)}(x) = c x^{\alpha}, \\ F_A^{(U)}(x) = 1 - d/x^{\beta}, \end{cases}$$

and

$$f_{A_c}(x) = \begin{cases} f_{A_c}^{(L1)}(x) + f_{A_c}^{(L2)}(x), & 0 \leqslant x \leqslant h', \\ f_{A_c}^{(U1)}(x) + f_{A_c}^{(U2)}(x), & h' < x < \infty, \end{cases}$$

$$F_{A_c}(x) = \begin{cases} F_{A_c}^{(L1)}(x) + F_{A_c}^{(L2)}(x), \\ F_{A_c}^{(U1)}(x) + F_{A_c}^{(U2)}(x), \end{cases}$$

where $h' = b^2/Ma^2$ and for example

$$f_{A_c}^{(L_1)}(x) = (\tfrac{1}{2})\alpha_1 w_1 x^{\alpha_1/2-1}, \quad F_{Ac}^{(L1)}(x) = w_1 x^{\alpha_1/2},$$

and

$$f_{A_c}^{(U2)}(x) = (\tfrac{1}{2})\beta_1 w_4 / x^{\beta_1/2+1}, \quad F_{A_c}^{(U2)}(x) = 1 - w_4/x^{\beta_1/2},$$

then it can be shown that

$$\begin{aligned}
P_f &= P(A_c \leqslant A)\\
&= \frac{2\alpha}{2\alpha+\alpha_1} F_A^{(L)}(h) F_{A_c}^{(L1)}(h) + \frac{2\alpha}{2\alpha+\beta_2} F_A^{(L)}(h) F_{A_c}^{(L2)}(h)\\
&\quad + \frac{\alpha_1}{\alpha_1 - 2\beta}\left(1 - F_A^{(U)}(h')\right) F_{A_c}^{(L1)}(h') + \frac{\beta_2}{\beta_2 - 2\beta}\left(1 - F_A^{(U)}(h')\right) F_{A_c}^{(L2)}(h')\\
&\quad - \frac{2\beta}{\alpha_1 - 2\beta}\left(1 - F_A^{(U)}(h)\right) F_{A_c}^{(L1)}(h) - \frac{2\beta}{\beta_2 - 2\beta}\left(1 - F_A^{(U)}(h)\right) F_{A_c}^{(L2)}(h)\\
&\quad - \frac{\alpha_2}{\alpha_2 + 2\beta}\left(1 - F_A^{(U)}(h')\right)\left(1 - F_{A_c}^{(U1)}(h')\right) - \frac{\beta_1}{\beta_1 + 2\beta}\left(1 - F_A^{(U)}(h')\right)\left(1 - F_{A_c}^{(U2)}(h')\right)
\end{aligned}$$

5. Conclusion

A parametric solution is obtained for the reliability calculation involving a simple stress-strength model of failure. The importance of tail consideration is pointed out and two distributions, namely Power and Pareto are proposed to describe their behavior. A distribution formed by combining the Power and Pareto models is introduced and it's application to reliability calculation is described and is demonstrated. It is concluded that the Power and Pareto models can provide solutions for both cases, namely when complete distributions are of interest or where only their tails are of importance. It is also discussed that, using this approach one will overcome some of the difficulties encountered in reliability calculation.

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